

ON THE HAHN-BANACH THEOREM

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I love the Hahn-Banach theorem. I love it the way I love *Casablanca* and the Fontana di Trevi. It is something not so much to be read as fondled. What is "the Hahn-Banach theorem?" Let f be a continuous linear functional defined on a subspace M of a normed space X . Take as the Hahn-Banach theorem the property that f can be extended to a continuous linear functional on X without changing its norm. Innocent enough, but the ramifications of the theorem pervade functional analysis and other disciplines (even thermodynamics!) as well. Where did it come from? Were Hahn and Banach the discoverers? The axiom of choice implies it, but what about the converse? Is Hahn-Banach equivalent to the axiom of choice? (No.) Are Hahn-Banach extensions ever unique? They are in more cases than you might think, when the unit ball of the dual is "round," as for ℓ_p with $1 < p < \infty$, for example, but not for ℓ_1 or ℓ_∞ . Instead of a linear functional, suppose we substitute a normed space Y for the scalar field and consider a continuous linear map $A : M \rightarrow Y$. Can A be continuously extended to X with the same norm? Well, sometimes. Unsurprisingly, it depends on Y , more specifically, on the "geometry" of Y : If the unit ball of Y is a "cube," as for $Y = (\mathbf{R}^n, \|\cdot\|_\infty)$ or $Y = \text{real } \ell_\infty$, for example, then for any subspace M of any X , any bounded linear map $A : M \rightarrow Y$ can be extended to X with the same norm. This is not true if $Y = (\mathbf{R}^n, \|\cdot\|_p)$, $n > 1$, for $1 < p < \infty$, despite the topologies being identical. The cubic nature of the unit ball does not suffice, however—if $Y = c_0$, the extendibility dies. This article traces the evolution of the analytic form as well as subsequent developments up to 2004.

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