10. Complexity

There are many types of languages we have not discussed so far. Languages defined by complexity conditions have interesting applications to group theory. We mention two examples.

10.1. **Two–way automata.** A two–way automaton has a read–only input tape, which can be traversed in both directions, and a separate work tape. Two–way

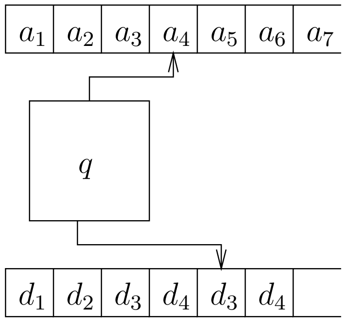

Figure 1. A 2-way automaton.

automata are equivalent to Turing machines in computational power, so let require that on an input of length n the work tape be of fixed length equal to the integer part of $\log(1+n)$. Languages which are accepted by machines of this type are said to have space complexity $\log n$.

Theorem 10.1. *The word problem of a linear group has space complexity* $\log n$.

Proofs of this and related results are given in [2, 3, 4]

Since free groups are linear, this theorem guarantees that the word problem of a free group can be solved in log–space. Using a pushdown automaton to solve that word problem directly by storing the free reduction of the input on the stack requires linear space. When the automaton is in the middle of processing the input $a^n a^{-n}$ the stack contains a^n, which is half as long as the input.

10.2. **Groups with word problem in NP.** Birget, Olshanskii, Rips and Sapir have proved a striking complexity–theoretic analog of the Higman Embedding Theorem. They show that a finitely generated group G has word problem in NP if and only if G is a subgroup of a finitely presented group with polynomial Dehn function [1].

References

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