11. Rational Relations

A relation $\rho : S \to T$ is a subset of $S \times T$. We define $\rho(s) = \{t \mid (s,t) \in \rho\}$; and $\rho(S')$ defined similarly for $S' \subset S$. As usual $\rho(s) = t$ means $\rho(s) = \{t\}$. If $\rho : S \to S$, we say ρ is a relation on S. A relation $\rho : M_1 \to M_2$ between monoids is a rational relation if it is a rational subset of $M_1 \times M_2$.

Exercise 11.1. *If M_1 is a finitely generated monoid, then any homomorphism $f : M_1 \to M_2$ is a rational relation.*

Exercise 11.2. *Every rational subset $R \subset M$ of a monoid M determines a rational relation*

$$\rho_R(m) = \begin{cases} m & \text{if } m \in R \\ \phi & \text{otherwise.} \end{cases}$$

The following lemma summarizes our progress and includes a few additional results, which are not difficult to prove.

Lemma 11.3. *Let M_1 and M_2 be monoids.*

1. *If $\rho : M_1 \to M_2$ is a partial homomorphism whose domain is a finitely generated submonoid of M_1, then ρ is rational.*
2. *If $R \subset M_1$ is rational, then so is $\rho_R = \{(m,m) \mid m \in R\}$.*
3. *If $\rho : M_1 \to M_2$ is rational, then so is the relation $\rho^{-1} : M_1 \to M_2$ defined by $\rho^{-1} = \{(m_2, m_1) \mid (m_1, m_2) \in \rho\}$.*
4. *If $L_1 \subset M_1$ and $L_2 \subset M_2$ are rational, then $\rho = L_1 \times L_2$ is rational too.*

The Reidemeister–Schreier rewriting process for a subgroup H of finite index in a group G is another example of a rational relation. Let $S = S^{-1}$ be a finite set of generators for G, and make the coset diagram of G/H into a finite automaton $\mathcal{A}$ accepting H by taking H as initial and terminal vertex. Pick a directed spanning tree Γ of $\mathcal{A}$ rooted at H. Edges of $\mathcal{A} - \Gamma$ determine a set of generators of H. Add a second coordinate to each edge label of $\mathcal{A}$. For edges in Γ the second coordinate is 1, and for edges in $\mathcal{A} - \Gamma$ the coordinate is the the corresponding generator of H. The rational relation $\rho : G \to H$ accepted by the modified version of $\mathcal{A}$ rewrites words in S which represent elements of H as products of the generators for H.

Unfortunately the composition of rational relations is not necessarily rational.

Exercise 11.4. *Define a homomorphism $\rho : (a+b)^* \to Z$ by $\rho(a) = 1, \rho(b) = -1$. Observe that ρ and ρ^{-1} are rational relations. Show that $\rho^{-1} \circ \rho$ is not.*

Theorem 11.5. *If $\rho : M \to \Sigma^*$ and $\rho : \Sigma^* \to M'$ are rational relations, so is $\tau \circ \rho$.*

Proof. Pick sets of generators A and A' for M and M' with both sets of generators containing the identity. Let $\Sigma_1 = \Sigma + \lambda$. We know that ρ is accepted by an automaton $\mathcal{A}$ over $A \times \Sigma_1$ and τ is accepted by $\mathcal{A}'$ over $\Sigma_1 \times A'$. We may assume that each vertex of $\mathcal{A}$ has a loop with label $(1, \lambda)$, and each vertex of $\mathcal{A}'$ has one with label $(\lambda, 1)$. Adding such loops does not alter the relations accepted by these automata. We will see that the automaton $\mathcal{B}$ over $A \times A'$ defined as follows accepts $\rho \circ \tau$.

1. The vertices of $\mathcal{B}$ are ordered pairs (p, p') of vertices from $\mathcal{A}$ and $\mathcal{A}'$ respectively.
2. There is an edge from (p, p') to (q, q') with label (a, c) if and only if for some $b \in \Sigma_1$ there are edges from p to q with label (a, b) and from p' to q' with label (b, c) in $\mathcal{A}$ and $\mathcal{A}'$.

3. The initial vertex is (p_0, p'_0) where p_0 and p'_0 are the initial vertices of $\mathcal{A}$ and $\mathcal{A}'$.
4. The set of terminal vertices is the product of the terminal vertices of $\mathcal{A}$ and $\mathcal{A}'$.

It is straightforward to show by induction on path length that there is a path in $\mathcal{B}$ from (p, p') to (q, q') with label (u, w) if and only if the following conditions hold.

1. For some word $v \in \Sigma^*$ there is a path in $\mathcal{A}$ from p to q with label (u, v) and a path in $\mathcal{A}'$ from p' to q' with label (v, w).
2. In both path labels v is expressed as a product of elements of Σ_1 in exactly the same way.

It follows immediately that if $\mathcal{B}$ accepts (u, w), then $\tau \circ \rho(u) = w$. For the converse suppose $\rho(u) = v$ and $\tau(v) = w$. There are successful paths in $\mathcal{A}$ and A' with labels (u, v) and (v, w). In the label of each path v occurs as a sequence of elements of Σ_1, but the two sequences can differ by insertion and deletion of λ's. Using the added loops, we can adjust the successful paths to make the two sequences identical. □

A binary relation $\rho : \Sigma^* \to \Delta^*$ between finitely generated free monoids is called a transduction. Rational transductions are useful object. For example if $L = \{u\$v\$w\}$ is context–free where u, v, w are words over an alphabet which does not contain \$, what kind of language is $\{u \mid \exists v, w \text{ such that } u\$v\$w \in L\}$? By writing down the right rational transduction we can see immediately that this language is context–free.

Corollary 11.6. *Rational transductions are closed under composition.*

Corollary 11.7. *If $\rho : \Sigma^* \to M$ is a rational relation and $L \subset \Sigma^*$ is rational, then $\rho(L)$ is rational too.*

Proof. $\rho(L)$ is the projection of $\rho \circ \rho_L$ from $\Sigma^* \times M$ to M. Apply Lemma 11.3. □

In the following theorem an arbitrary rational relation is decomposed into a composition of relations known to be rational by Lemma 11.3.

Theorem 11.8 (Nivat's Theorem)**.** *If $\rho : M_1 \to M_2$ is a rational relation, then there is a free monoid Δ^* with homomorphisms $\pi_i : \Delta^* \to M_i$ and a rational subset $L \subset \Delta^*$ such that $\rho = \pi_2 \circ \rho_L \circ \pi_1^{-1}$.*

Proof. Let ρ be accepted by an automaton $\mathcal{A}$ over $M_1 \times M_2$, and take Δ to be an alphabet with one letter for each edge of $\mathcal{A}$. Define $\tilde{\mathcal{A}}$ to be $\mathcal{A}$ with each edge e labeled by its letter $d_e \in \Delta$.

$\mathcal{A}$ and $\tilde{\mathcal{A}}$ are the same automaton except for edge labels. Each edge e of $\mathcal{A}$ has a label (m_1, m_2). For $i = 1, 2$, let $\pi_i : \Delta^* \to M_i$ be the homomorphism determined by $\pi_i(d_e) = m_i$. Take L to be the rational language accepted by $\tilde{\mathcal{A}}$. It is straightforward to check that the desired result holds. □

Corollary 11.9. *Abstract families of languages are closed under rational transduction.*