A REMARK ABOUT COMBINGS OF GROUPS

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Section 1. Introduction

In the last several years a remarkable interplay between geometry, group theory, and the theory of formal languages has led to developments including the introduction of automatic groups [Ep+], hyperbolic groups [Gro], and geometric and language-theoretic characterizations of finitely generated virtually free groups [MS1] [MS2]. A prime example of such interplay is the study of combings, which were introduced in [Ep+, Chapter 3.6].

A *combing* is simply a choice of normal form for group elements relative to a fixed finite set of generators, but typically the set of words in normal form is required to satisfy language theoretic constraints when viewed as a formal language over the alphabet of generators, and geometric constraints when interpreted as a collection of paths in the Cayley graph of the group. For example, a group G is said to be automatic [Ep+] if for some finite set of generators it has a set of normal forms which is a *regular language* and has the property that the paths determined by normal forms for nearby elements are uniformly close, i.e., there exists a constant $K > 0$ such that points travelling at unit speed along two such paths which end a distance 1 apart stay K-close; this last condition is called the synchronous fellow traveller property. If these two conditions hold for one finite set of generators then they hold for all.

Language theoretic and geometric conditions afford new ways of looking at finitely generated groups, and it is of interest to consider how various classes of groups may be characterized in terms of the combings which they admit. A well known open question in the study of automatic groups is whether or not one needs to include the language theoretic constraint of regularity in the definition. Might it be the case that every group which admits a set of normal forms satisfying the synchronous fellow traveller property is automatic, i.e., admits a regular set of normal forms satisfying the synchronous fellow traveller property? It seems highly unlikely that this will turn out to be the case, and various possible counterexamples have been proposed, including the three dimensional Heisenberg group $\mathcal{H}_3$, and irreducible uniform lattices in $Sl_2(\mathbb{R}) \times Sl_2(\mathbb{R})$. However, despite much effort, this question remains unresolved. What is known is that $\mathcal{H}_3$ provides a counterexample to the analogous question involving the asynchronous fellow traveller property. This

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result was first established by [Ep+] together with [Bri] and has been sharpened in [BG].

In this note we return to the spirit of the original question. We give a simple example of a class of groups defined in terms of the combings which they admit with respect to an arbitrary set of generators, and show that in this simpler situation it is not necessary to include the language theoretic constraint as part of the definition.

Theorem A. *The following conditions are equivalent for a finitely generated group G.*

- (i) *With respect to some finite set of generators, G has a broomlike combing.*
- (ii) *With respect to every finite set of generators, G has a regular broomlike combing.*
- (iii) *G is virtually free, i.e., G has a free subgroup of finite index.*

A combing is said to be *broomlike* if it satisfies the simple geometric condition described in (2.5) below. Other geometric characterizations of virtually free groups are contained in [MS1], [MS2] and [W], and a further variation on this theme was discovered by G. Mess and M. Shapiro (unpublished). The proof which we shall give of Theorem A relies on the result of [MS1] (together with [Dun]) that a finitely generated group is virtually free if and only if the set of words defining the identity is a context free language.

Section 2. Preliminary Definitions

Formal Languages. For any finite set Σ, denote by Σ^* the free monoid over Σ. The elements of Σ^* are words over the alphabet Σ, i.e., finite sequences of members of Σ. The length of a word w is denoted $|w|$. The identity element of Σ^* is the empty word ϵ, and multiplication of words is by concatenation. Σ is the unique set of free generators of Σ^*. If $L, M \subset \Sigma^*$, then the product of L and M is $LM = \{ w \mid w = xy, x \in L, y \in M \}$.

A formal language over the alphabet Σ is just a subset $L \subset \Sigma^*$. See Hopcroft and Ullman [HU] for an introduction. We mention two classes of languages, regular languages and the larger class of context free languages. Regular languages are the closure of finite subsets of free monoids under union, product, and generation of submonoid. Context free languages are those accepted by pushdown automata.

Definition 2.1. *A pushdown automaton $\mathcal{P} = (Q, \Sigma, \Delta, \delta, q_0, Z_0)$ consists of*

- (i) *A finite nonempty set of states Q;*
- (ii) *An input alphabet Σ;*
- (iii) *A stack alphabet Ω;*
- (iv) *An initial state $q_0 \in Q$;*
- (v) *A start symbol $Z_0 \in \Omega$;*
- (vi) *A mapping δ from $Q \times \Omega \times (\Sigma \cup \{\epsilon\})$ to finite subsets of $Q \times \Omega^*$.*

We will use Z to denote an element of Ω and ω for a word over Ω; in particular ωZ is a nonempty word. Usually a PDA is construed as a kind of computer; in [MS2] however, a slight variation on the unwieldy Definition 2.1 serves as a tool for constructing certain infinite labelled directed graphs, among them the Cayley diagrams of virtually free groups.

Definition 2.2. *A configuration of a pushdown automaton $\mathcal{P}$ is an element of $Q \times \Omega^*$.*

A PDA $\mathcal{P}$ acts as a computer by starting in configuration (q_0, Z_0) and moving from one configuration to another according to the mapping δ. (Thus δ is the program of this computer.) More precisely suppose $\mathcal{P}$ is in the configuration $(q, \omega Z)$ and a is the next letter of its input. If $(p, \omega') \in \delta(q, Z, a)$, then $\mathcal{P}$ may move to the configuration $(p, \omega\omega')$, and continue its computation with the input letter following a. If $(p, \omega') \in \delta(q, Z, \epsilon)$, then $\mathcal{P}$ may make the same move while keeping a as the next input letter following a. In the first case we say that $\mathcal{P}$ reads the letter a of its input, and in both cases we say that $\mathcal{P}$ pops Z from its stack and pushes ω' onto its stack.

If there is only one move that $\mathcal{P}$ may make from a given configuration, then it makes that move. If $\mathcal{P}$ has a choice of moves, it may make any one of them. In this case $\mathcal{P}$ is called nondeterministic. It is also clear that $\mathcal{P}$ might get stuck; for example it cannot move if its stack is empty.

The language accepted by a PDA (by empty stack) is the set of inputs for which there is a sequence of moves starting from (q_0, Z_0) and ending at any configuration (q, ϵ) during which all the input is read. In the nondeterministic case there may also be other successful or unsuccessful sequences of moves starting from (q_0, Z_0). A language is context free if and only if it is accepted by a PDA [HU, Chapter 5].

Given a PDA $\mathcal{P}$ as in Definition 2.1, one can define in a straightforward way a mapping $\hat{\delta}$ from $Q \times \Omega \times \Sigma^*$ to subsets of $Q \times \Omega^*$ such that $\hat{\delta}(q, \omega, w)$ is the set of all configurations which can be reached starting from configuration (q, ω) and reading input w. The language accepted by $\mathcal{P}$ is the set of all w such that $\hat{\delta}(q_0, Z_0, w)$ contains a configuration (q, ϵ). Definition 2.1 may be viewed as just a way of defining a certain class of functions $\hat{\delta}$.

Combings of Groups. All groups under discussion are understood to be finitely generated.

Definition 2.3. *A choice of generators for a group G is a surjective monoid homomorphism $\mu : \Sigma^* \twoheadrightarrow G$. (Given $w \in \Sigma^*$, we often write $\overline{w}$ in place of $\mu(w)$.)*

We denote letters in Σ by $a, b, \ldots$, words in Σ^* by $u, v, w \ldots$, and elements of G by $g, h, \ldots$. Also we let Σ^{-1} be a set of formal inverses to Σ and extend μ in the obvious way to $\mu : (\Sigma \cup \Sigma^{-1})^* \to G$. We extend the map $a \to a^{-1}$, $a \in \Sigma$, to $w \to w^{-1}$, $w \in (\Sigma \cup \Sigma^{-1})^*$ so that $\mu(w^{-1}) = \mu(w)^{-1}$.

Any choice of generators determines a left invariant word metric on G.

Definition 2.4. $d_\mu(g, h) = \min\{|w| \mid w \in (\Sigma \cup \Sigma^{-1})^*,\ \mu(w) = g^{-1}h\}$.

When there is no danger of ambiguity, we shall write d rather than d_μ. It is straightforward to check that d is indeed a metric on G and that d is left invariant in the sense that $d(gh_1, gh_2) = d(h_1, h_2)$. The metrics determined by different choices of generators are Lipschitz equivalent; i.e., if $\nu : \Lambda \to G$ is another choice of generators, then for some constant C and all $g, h \in G$, $1/Cd_\mu(g, h) \leq d_\nu(g, h) \leq Cd_\mu(g, h)$.

Definition 2.5. *Let $\mu : \Sigma^* \twoheadrightarrow G$ be a choice of generators and L a language over Σ.*

(i) *L is a combing of G if it projects bijectively to G.*
(ii) *If the combing L lies in a class of formal languages $\mathcal{A}$, then L is an $\mathcal{A}$-combing.*
(iii) *If there is a constant k such that any two words w, v in the combing L with $d(\overline{w}, \overline{v}) = 1$ have the form $w = uw'$, $v = uv'$ with $|w'| + |v'| \leq k$, then the combing L is broomlike.*

Remark. If one views words $w \in \Sigma^*$ as paths in the Cayley graph of G starting at $1 \in G$, (see, e.g., [Ep+]), then condition (iii) can be interpreted as saying that the combing paths to each pair of adjacent elements share a common initial segment from which they are obtained by appending paths of uniformly bounded length. For example, if the Cayley graph of G contains a spanning tree which is quasi-isometrically embedded, then the geodesics in this tree that connect each element to the identity form a broomlike combing. Indeed it is a consequence of Theorem A that a group admits a broomlike combing if and only if it admits a "treelike" combing of this type.

The proof of the following lemma is clear.

Lemma 2.6. *If L is a broomlike combing of G, then after replacement of one word in L by another word representing the same group element, L is still a broomlike combing, perhaps with a larger constant k.*

Section 3. The Proof of Theorem A

That condition (ii) implies (i) is trivial. To show that (iii) implies (ii) we consider a group G with a free subgroup of finite index F, and fix a set R of right coset representatives for F in G. We shall restrict our attention to a convenient choice of generators $\mu : \Sigma \to G$. The proof in the general case follows from the observation that if $\nu : \Lambda \to G$ is an arbitrary set of generators for G and $L \subseteq \Sigma^*$ is a regular broomlike combing of G, then one obtains a broomlike combing $L' \subseteq \Lambda^*$ as the image of L under any monoid homomrophism $f : \Sigma^* \to \Lambda^*$ with $\mu = \nu \circ f$. It is clear that L' is a broomlike combing, and as the class of regular languages is closed under monoid homomorphisms [HU], L' is regular too.

We construct the desired generating set Σ as follows. Take T to be a set of free semigroup generators for F and $\Sigma = R \cup T$. Let $\tau : \Sigma^* \twoheadrightarrow G$ be the choice of generators induced by the inclusion of Σ into G. Recall that τ extends to $\Sigma \cup \Sigma^{-1}$. Let L denote the set of all words wr such that w is a freely reduced word over $T \cup T^{-1}$ and $r \in R$. We claim that L is a regular broomlike combing of G. Regularity is easy as L may be constructed from Σ using the operations mentioned at the beginning of Section 2. Likewise it is clear that τ maps L bijectively to G whence L is a combing.

To check that L is broomlike suppose $d(\overline{w}, \overline{v}) = 1$ for $w, v \in L$. We have $w = w_1 r_1$ and $v = w_2 r_2$ for some $r_i \in R$ and w_i freely reduced over $T \cup T^{-1}$. By assumption $\overline{w_1 r_1} = \overline{w_2 r_2 a}$ for some $a \in \Sigma$. From the structure of G it follows that $\overline{r_2 a} = \overline{v r_1}$ for some word v over $T \cup T^{-1}$. Because there are only a finite number of possibilities for r_2 and a, there is a constant k' such that $|v| \leq k'$. Finally $w_1 v$ must freely reduce to w_2 whence it follows that the condition of Definition 2.5(iii) is satisfied with $k = k' + 2$.

It remains to show that (i) implies (iii). By the result of Muller and Schupp mentioned at the end of Section 1 it suffices to construct a PDA which accepts $L = \{w \mid w \in \Sigma^*, \overline{w} = 1\}$. The main point is the following.

Lemma 3.1. *If G has a broomlike combing L with respect to a choice of generators $\mu : \Sigma^* \twoheadrightarrow G$, then there is for each $a \in \Sigma \cup \Sigma^{-1}$ a finite set of words $W(a) \subset (\Sigma \cup \Sigma^{-1})^*$ such that a word $w = a_1 \dots a_n \in (\Sigma \cup \Sigma^{-1})^*$ represents the identity in G if and only if it is possible to replace each a_i by a word of $W(a_i)$ so that the resulting word freely reduces to ϵ.*

Proof. Let k be the constant from Definition 2.5(iii). Take $W(a)$ to consist of all words in $(\Sigma \cup \Sigma^{-1})^*$ which project to $\overline{a}$ and have length at most k. Clearly each $W(a)$ is finite; and if any replacement of letters in $w = a_1 \dots a_n$ by words in $W(a_i)$ produces a word which freely reduces to ϵ, then $\overline{w} = 1$.

For the converse suppose that $w = a_1 \dots a_n$ with $\overline{w} = 1$. By Lemma 2.6 we may assume $\epsilon \in L$ represents $1 \in G$. Let $v_0 = \epsilon$, and for $0 \leq i \leq n$ let v_i be the unique element of L such that $\overline{v_i} = \overline{a_0 \dots a_i}$. Note that $v_0 = v_n = \epsilon$. Because $\overline{v_{i-1} a_i} = \overline{v_i}$ for $1 \leq i \leq n$, Definition 2.5(iii) allows us to decompose the words v_{i-1} and v_i in the free monoid Σ^* as $v_{i-1} = u_i x_i$, $v_i = u_i y_i$, where $|x_i| + |y_i| \leq k$, and $x_i^{-1} y_i \in W(a_i)$. Substitution of $x_i^{-1} y_i$ for a_i, yields

$$\begin{aligned}(x_1^{-1}y_1)(x_2^{-1}y_2)\dots(x_n^{-1}y_n) &\sim x_1^{-1}(u_1^{-1}u_1)y_1x_2^{-1}(u_2^{-1}u_2)y_2\dots x_n^{-1}(u_n^{-1}u_n)y_n \\ &\sim (x_1^{-1}u_1^{-1})(u_1y_1)(x_2^{-1}u_2^{-1})(u_2y_2)\dots(x_n^{-1}u_n^{-1})(u_ny_n) \\ &\sim v_0^{-1}v_1v_1^{-1}v_2\dots v_{n-1}^{-1}v_n \\ &\sim \epsilon\end{aligned}$$

where $\sim$ denotes free equivalence. Thus $x_1^{-1}y_1x_2^{-1}y_2\dots x_n^{-1}y_n$ freely reduces to ϵ. In other words it represents the identity in the free group over Σ. □

Now it is easy to complete the proof of Theorem A by constructing a pushdown automaton $\mathcal{P}$ which given input $a_1 \dots a_n$ will for each i, $1 \leq i \leq n$ pick $w_i \in W(a_i)$ and push it onto the stack in such a way that the stack contains the free reduction of $w_1 \dots w_i$. If after processing the input the stack contains only the start symbol, then $\mathcal{P}$ accepts the input by popping the start symbol from the stack. The states of $\mathcal{P}$ will be all freely reduced words in $\Sigma \cup \Sigma^{-1}$ of length at most k, and $\mathcal{P}$ will pick $w_i \in W(a_i)$ by moving non-deterministically to state w_i. It is straightforward to check that the following definition works.

Definition 3.2. *Define the PDA $\mathcal{P} = (Q, \Sigma, \Delta, \delta, q_0, Z_0)$ as follows.*

- (i) *The set of states Q consists of all words in $\Sigma \cup \Sigma^{-1}$ of length at most k;*
- (ii) *The input alphabet is $\Sigma \cup \Sigma^{-1}$;*
- (iii) *The stack alphabet is $\Sigma \cup \Sigma^{-1}$ together with another symbol Z_0;*
- (iv) *The initial state is ϵ;*
- (v) *The start symbol is Z_0;*

(vi) *The mapping δ from $Q \times \Sigma \cup \Sigma^{-1} \cup \{Z_0\} \times (\Sigma \cup \Sigma^{-1} \cup \{\epsilon\})$ to finite subsets of $Q \times (\Sigma \cup \Sigma^{-1} \cup \{Z_0\})^*$ is given by*

$$\delta(q, Z, a) = \begin{cases} W(a) \times \{Z\}, & \text{if } q = \epsilon \text{ and } a \in \Sigma \cup \Sigma^{-1}; \\ (a_2 \dots a_m, \epsilon), & \text{if } q = a_1 \dots a_m, m \geq 1,\ Z = a_1^{-1}, \text{ and } a = \epsilon; \\ (a_2 \dots a_m, Za_1), & \text{if } q = a_1 \dots a_m, m \geq 1,\ Z \neq a_1^{-1}, \text{ and } a = \epsilon; \\ (\epsilon, \epsilon), & \text{if } q = a = \epsilon \text{ and } Z = Z_0; \\ \text{The empty set}, & \text{otherwise.} \end{cases}$$

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